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A version of Rolle's theorem and applications

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Abstract. This paper deals with global injectivity of vector fields defined on euclidean spaces. Our main result establishes a version of Rolle's Theorem under generalized Palais-Smale conditions. As a consequence of this, we prove global injectivity for a class of vector fields defined on n-dimensional spaces.
Keywords: Rolle's theorem, minimax methods, global injectivity.

1. Introduction

Results on global univalence of vector fields have been the object of intense research in recent years. In several branches of Mathematics important questions are related to these results. We refer the reader to [**9,7,3**] for further references and connections with other problems.

In this work, our main purpose is to establish, via minimax methods, new versions of Rolle's Theorem, providing further sufficient conditions to ensure global univalence for locally injective vector fields.

Given a real Banach space E and a functional $f : E \to \mathbb{R}$ of class C^1 satisfying a generalized version of the compactness condition introduced by Palais-Smale [**11**], we relate the existence of critical points for f with the topology of the level surfaces $S_c(f) = f^{-1}(\{c\})$, $c \in \mathbb{R}$.

In our basic theorem, we show that either $S_c(f)$ is path-connected or $f : E \to \mathbb{R}$ possesses a critical point. This result can be easily seen as a version of Rolle's Theorem for domains with dimension greater than one.

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We also provide a minimax characterization of the possible critical levels of the functional, relating Rolle's Theorem with the famous Mountain Pass Theorem of Ambrosetti-Rabinowitz [**1**].

We generalize our basic theorem by proving the existence of critical levels when $S_c(f)$ is not homologically trivial. This relates our results with the notion of linking and generalizations of the Mountain Pass Theorem.

As a consequence of our basic theorem, we establish a version of Rolle's Theorem for vector fields of class C^1 on $\mathbb{R}^2$. Using this result, we prove global injectivity for a class of locally injective vector fields on the plane.

In this article, we also prove a version of Rolle's Theorem for vector fields of class C^2 on higher dimensions. To derive such theorem, we establish a deformation lemma based on the minor determinants of the Jacobian matrix of the vector field. The global univalence of those vector fields is also studied. Unlike most of the known results, to obtain global univalence for a local diffeomorphism on $\mathbb{R}^n$, we assume an extra hypothesis on only (n-1)-coordinates of the vector field.

In [**10**], Rabier used a deformation result and arguments of differential topology to generalize Hadamard Theorem on global diffeomorphism for euclidean spaces. We notice that in [**10**], the author considers a notion of admissible flow which is closely related to our generalization of the Palais-Smale condition. In [**4**], Katriel studied global homeomorphism theorem for certain topological and metric spaces by proving two versions of the Mountain Pass Theorem without assuming that the functional is of class C^1.

We should also mention that new versions of Rolle's Theorem have been recently obtained by Khovanskii and Yakovenko [**5**] (See also the references therein). Considering analytic functions on the complex plane, in [**5**], the authors are able to relate the number of zeros of the function with the number of zeros of its derivative.

In [**6**] is proved that a C^1-mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is globally one-to-one provided that:(i) for each $x \in \mathbb{R}^2$ $det f'(x) \neq 0$ and (ii) there exist linearly independent vectors v_i $(i = 1, 2)$ in $\mathbb{R}^2$ such that 0 does not belong to the convex hull $\{f'(x)v_i; x \in \mathbb{R}^2\}$, $i = 1, 2$.

The main result proved in [**3**] is related to the Conjecture on Global Asymptotic Stability (or Marcus-Yamabe Conjecture) in $2D$. It says the following: A C^1−mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is globally one-to-one provided that $f(0) = 0$ and all the eigenvalues of $f'(x)$ have negative real parts, for every $x \in \mathbb{R}^2$.

The article is organized in the following way: In section 2, we present the necessary preliminary results. There, we recall a version of the Mountain Pass Theorem after stating a deformation lemma for a generalization of Palais-Smale condition. In section 3, we prove our basic result and its generalization. In section 4, we prove Rolle's Theorem on $\mathbb{R}^2$ and study its applications. In section 5, after presenting a version of the deformation lemma for euclidean vector fields, we state a version of Rolle's Theorem on $\mathbb{R}^n$, $n \geq 3$. There, we also study the question of injectivity for dimension greater than two. Finally, in section 6, we prove the deformations results stated in sections 2 and 5.

We are thankful to the referee for the helpful suggestions and comments. He has also provided the reference [**4**] which was not known by the authors. Following his suggestion, it is our intention to apply the argument employed in [**4**] to derive topological versions of the results obtained in this article.

2. Preliminaries

Let E be a real Banach space and consider $f : E \to \mathbb{R}$ a functional of class C^1. Given $c \in \mathbb{R}$, we define $S_c(f) = f^{-1}(\{c\})$, the associated level surface of f. By $B_\rho(u)$ and $\partial B_\rho(u)$ we denote, respectively, the closed ball of radius ρ centered at u and its boundary. By K we denote the set of critical points of f. Given $d \in \mathbb{R}$, we set $f^d = \{u \in E \mid f(u) \leq d\}$, $f_d = \{u \in E \mid f(u) \geq d\}$ and $K_d = \{u \in E \mid f(u) = d,\ f'(u) = 0\}$.

Consider Λ, the family of functions $\phi : (0, \infty) \to (0, \infty)$ which are nonincreasing, locally Lipschitz continuous and satisfy $\int_0^\infty \phi(t)\, dt = \infty$.

The following version of Palais-Smale condition is assumed:

Definition 2.1. *Given $f \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$, we say that f satis-*

fies the Generalized Palais-Smale condition (with respect to $\phi \in \Lambda$*) at level* $c \in \mathbb{R}$*, denoted* $(GPS)_c$*, if every sequence* $(u_m) \subset E$ *satisfying* $f(u_m) \to c$ *and* $\|f'(u_m)\|/\phi(\|u_m\|) \to 0$*, as* $m \to \infty$*, possesses a converging subsequence. If* f *satisfies* $(GPS)_c$ *for every* $c \in \mathbb{R}$*, we just say that it satisfies* (GPS)*.*

We note that when $\phi(t) \equiv 1$ and $\phi(t) = 1/(1+t)$, we have, respectively, the usual Palais-Smale condition [**11**], denoted by (PS), and its generalization due to Cerami [**13**]. In order to obtain a deformation lemma for (GPS) condition, we shall need to make a restriction on the topological structure of the critical points of the functional.

Definition 2.2. *We say that a component of a set of* E *is trivial if it has only a point. Given* $f \in C^1(E, \mathbb{R})$*, we say that* $c \in \mathbb{R}$ *is an admissible level if either* c *is a regular value of* f*, or the components of* K_c *are trivial and* c *is an isolated critical value of* f*.*

The following result is a sharper version under (GPS) condition of a deformation lemma due to Chang [**2**]. The proof is a variation of the original one and it will be given in section 6 (See also [**13**], for a related result).

Proposition 2.3. (**Deformation Lemma.**) *Suppose that* $f \in C^1(E, \mathbb{R})$ *and satisfies (GPS) with respect to some* $\phi \in \Lambda$*. Assume that* a *is the only possible critical value of* f *on the interval* $[a, b)$ *and that* a *is an admissible level. Then, there exists a continuous map*

$$\tau : [0,1] \times (f^b \setminus K_b) \to f^b \setminus K_b, \quad \textit{so that}$$

(i) $\tau(0, u) = u, \ \forall \ u \in f^b \setminus K_b$

(ii) $\tau(t, u) = u, \ \forall \ (t, u) \in [0,1] \times f^a$

(iii) $\tau(1, u) \in f^a, \ \forall \ u \in f^b \setminus K_b$

To provide a version of Rolle's theorem on spaces of dimension greater than two, in section 5, we establish a deformation lemma which depends on the minor determinants of the Jacobian matrix of the given vector field.

In what follows, we recall the Mountain Pass Theorem of Ambrosetti-Rabinowitz [**1**].

Theorem 2.4. *Let E be a real Banach space and suppose $f \in C^1(E, \mathbb{R})$ is a functional satisfying $f(0) = 0$ and (PS). Assume, f satisfies*

(i) *There exist $\rho > 0$ and $\alpha > 0$ such that*

$$f(u) \geq \alpha > 0, \ \forall\, u \in \partial B_\rho(0),$$

and

(ii) *There exists $e \in E \setminus B_\rho(0)$ such that $f(e) = 0$.*

Then, f has a critical value $c \geq \alpha$ characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)),$$

with

$$\Gamma = \{\gamma \in C([0,1], E) \mid \gamma(0) = 0, \ \gamma(1) = e\}.$$

We notice that Theorem 2.4 is a consequence of our basic result (See Remark 3.2-(iii)). In this article, we also establish a version of Theorem 2.4 for a setting where $\alpha = 0$ and the functional satisfies (GPS) with respect to some $\phi \in \Lambda$.

3. The Level Surface Theorem

In this section, we prove our basic result which is used in the sequel to establish a version of Rolle's Theorem on R^2. We also present another version of Theorem 2.4 and a result that relates the topology of the level surfaces of the functional and the existence of critical points.

Theorem 3.1. (**The Fundamental Theorem.**) *Let E be a real Banach space and suppose that $f : E \to \mathbb{R}$ is a functional of class C^1, satisfying (GPS) for some $\phi \in \Lambda$. Assume that $c \in \mathbb{R}$ is an admissible critical level of f and that u and v are two distinct points of S_c. Then, either*

(i) *u and v are in the same path-component of $S_c(f)$, or*

(ii) *f has a critical value $d \neq c$.*

Remark 3.2. (i) Theorem 3.1 can be seen as a version of Rolle's Theorem for functionals defined on real Banach spaces. (ii) As in Theorem 2.4, we obtain a minimax characterization of the possible values of d. Moreover, if $f \in C^1(E, \mathbb{R})$ and Theorem 3.1 fails, we have that f does not satisfy (GPS) in at least one of those possible values. (iii) Note also that

Theorem 3.1 may be seen as a generalization of Theorem 2.4 since the hypothesis of this last result implies that 0 and e are not in the same path-component of $S_0(f)$.

Before proving Theorem 3.1, we need to establish two preliminary results. Considering $u,\ v \in S_c(f)$, given in Theorem 3.1, we define

$$c_1 = \inf_{\gamma \in \Gamma_1} \max_{t\in[0,1]} f(\gamma(t)), \tag{3.1}$$

with

$$\Gamma_1 = \{\gamma \in C([0,1], E) \mid \gamma(0) = u,\ \gamma(1) = v\}. \tag{3.2}$$

As a consequence of Proposition 2.3, we have

Lemma 3.3. *Suppose c is an admissible value of f. Then, either $c_1 > c$ and c_1 is a critical value of f, or there exists $\gamma \in \Gamma_1$ such that*

$$\max_{t\in[0,1]} f(\gamma(t)) \le c. \tag{3.3}$$

Proof. First, we assume $c_1 > c$. Arguing by contradiction, we suppose that c_1 is not a critical value of f. From (GPS), there exists $0 < \epsilon < c_1 - c$ so that $K \cap f^{-1}([c_1 - \epsilon, c_1 + \epsilon] = \varnothing$.

We also have $\gamma \in \Gamma_1$ such that

$$\max_{t\in[0,1]} f(\gamma(t)) \le c_1 + \epsilon.$$

Applying Proposition 2.3, we obtain $\tau : [0,1] \times f^{c_1+\epsilon} \to f^{c_1+\epsilon}$ so that $\tau(t,u) = u$, for every $u \in f^{c_1-\epsilon}$, and $\tau(1, f^{c_1+\epsilon}) \subset f^{c_1-\epsilon}$. Then, $\tau(1,\gamma(t)) \in \Gamma_1$ and

$$\max_{t\in[0,1]} f(\tau(1,\gamma(t))) \le c_1 - \epsilon.$$

However, that contradicts the definition of c_1.

Next, we suppose $c_1 = c$. Since c is an admissible level of f, there exists $\epsilon > 0$ such that $K \cap f^{-1}([c-\epsilon, c+\epsilon]) = K_c$. Furthermore, we have $\gamma_1 \in \Gamma_1$ satisfying

$$\max_{t\in[0,1]} f(\gamma_1(t)) \le c + \epsilon.$$

As above, we may invoke Proposition 2.3 to obtain $\tau(t,u)$ satisfying (i)-(iii) with $a = c$ and $b = c + \epsilon$. It is not difficult to show that $\gamma(t) = \tau(1, \gamma_1(t))$ belongs to Γ_1 and satisfies (3.3). The lemma is proved. □

Now, we define

$$c_2 = \sup_{\gamma \in \Gamma_2} \min_{t \in [0,1]} f(\gamma(t)), \tag{3.4}$$

with

$$\Gamma_2 = \{\gamma \in C([0,1], f^c) \mid \gamma(0) = u, \ \gamma(1) = v\}. \tag{3.5}$$

Lemma 3.4. *Suppose that c is an admissible value of f and that $\Gamma_2 \neq \varnothing$. Then, either $c_2 < c$ and c_2 is a critical value of f, or there exists $\gamma \in \Gamma_2$ such that*

$$f(\gamma(t)) = c, \ \forall\ t \in [0,1]. \tag{3.6}$$

Proof. If $c_2 < c$ and c_2 is not a critical value of f, we take $0 < \epsilon < c - c_2$ and apply Proposition 2.3 on $-f$ to obtain $\tau : [0,1] \times f_{c_2-\epsilon} \to f_{c_2-\epsilon}$ such that $\tau(t,u) = u$, for every $(t,u) \in [0,1] \times f_{c+\epsilon}$, and $\tau(1, f_{c_2-\epsilon}) \subset f_{c_2+\epsilon}$. If we take $\gamma \in \Gamma_2$ so that

$$\min_{t \in [0,1]} f(\gamma(t)) \geq c_2 - \epsilon,$$

and consider $\hat{\gamma}(t) = \tau(1, \gamma(t))$, for $t \in [0,1]$, we get that $\hat{\gamma} \in \Gamma_2$ and

$$\min_{t \in [0,1]} f(\hat{\gamma}(t)) \geq c_2 + \epsilon.$$

But, this contradicts the definition of c_2. When $c_2 = c$, we apply Proposition 2.3 for $-f$ with $a = -c$, $b = -c + \epsilon$, and $\epsilon > 0$ sufficiently small. Then, we argue as above to obtain a path $\gamma \in \Gamma_2$ satisfying (3.6). The lemma is proved. □

Proof of Theorem 3.1. Suppose that u and v are not in the same path-component of $S_c(f)$. Considering c_1 and c_2 given by (3.1) and (3.4), respectively, we claim that at least one of those values is a critical value of f. Effectively, if we suppose otherwise, Lemma 3.2 implies $\Gamma_2 \neq \varnothing$. Consequently, by Lemma 3.3, we must have $\gamma : [0,1] \to S_c(f)$ such that $\gamma(0) = u$ and $\gamma(1) = v$. But, that contradicts the fact that u and v are not in the same path-component of $S_c(f)$. On the other hand, if f has no critical value $d \neq c$, then, necessarily, $c_1 = c_2 = c$ and u and v are in the same path-component of $S_c(f)$. Theorem 3.1 is proved. □

Remark 3.5. (i) It is clear from the proof that Theorem 3.1 holds if we assume $(GPS)_c$ for $c \in [c_1, c_2]$, where c_1 and c_2 are given by (3.1) and (3.4), respectively. (ii) Substituting f by $-f$ in Theorem 3.1, we obtain two other possible critical values of f:

$$c_3 = \sup_{\gamma \in \Gamma_1} \min_{t\in[0,1]} I(\gamma(t)), \tag{3.7}$$

with

$$\Gamma_1 = \{\gamma \in C([0,1], E) \mid \gamma(0) = u,\ \gamma(1) = v\}; \tag{3.8}$$

and

$$c_4 = \inf_{\gamma \in \Gamma_3} \max_{t\in[0,1]} I(\gamma(t)), \tag{3.9}$$

with

$$\Gamma_3 = \{\gamma \in C([0,1], f_c) \mid \gamma(0) = u,\ \gamma(1) = v\}. \tag{3.10}$$

Next, we extend to (GPS) condition a version of Theorem 2.4 proved in [**11**].

Theorem 3.6. *Let E be a real Banach space and suppose that $f \in C^1(E, \mathbb{R})$ is a functional satisfying $f(0) = 0$ and (GPS). Assume that f satisfies*

(i) *There exists $\rho > 0$ such that*

$$f(u) \geq 0,\ \forall\ u \in \partial B_\rho(0),$$

and

(ii) *There exists $e \in E \setminus B_\rho(0)$ such that $f(e) = 0$.*

Then, f has a critical value $c \geq 0$ characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{t\in[0,1]} f(\gamma(t)),$$

with

$$\Gamma = \{\gamma \in C([0,1], E) \mid \gamma(0) = 0,\ \gamma(1) = e\}.$$

Furthermore, if f does not have a critical point u on $\partial B_r(0)$, for some $0 < r \leq \rho$, then $c > 0$.

Proof. By Lemma 1.19 in [**12**], if f has no critical point $u \in \partial B_r(0)$, then there exist $\alpha > 0$ and a homeomorphism $\psi : E \to E$ such that $\psi(0) = 0$, $\psi(e) = e$ and $f(\psi(u)) \geq \alpha > 0$, for every $u \in \partial B_r(0)$. On that case, $c \geq \alpha > 0$ is a critical value of f by Theorem 3.1. The theorem is proved. □

Remark 3.7. As in Theorem 1.13 of [**12**], to get the homeomorphism used in the proof of Theorem 3.6, we need only a local version of Palais-Smale condition at level zero.

As our final result in this section, we state a natural generalization of Theorem 3.1 which relates the existence of critical points for the functional $f : E \to \mathbb{R}$ with the topology of the level surfaces of f. Given a topological space X, we denote by $\tilde{H}_k(X)$ the k-th reduced singular homology with integer coefficients.

Theorem 3.8. *Let E be a real Banach space and suppose that $f : E \to \mathbb{R}$ is a functional of class C^1 satisfying (GPS) for some $\phi \in \Lambda$. If $\tilde{H}_*(S_c(f))$ is not trivial for some $c \in \mathbb{R}$, then f possesses a critical value $d \neq c$.*

Proof. For $k \in \mathbb{N} \cup \{0\}$, we set $D_1^{k+1}(0) = \{x \in \mathbb{R}^{k+1} \mid \|x\| \leq 1\}$. Given $\gamma : S^k = \partial D_1(0) \to S_c(f)$, we denote by $[\gamma]$ the equivalence class of γ on $H_k(\tilde{S}_c(f))$. Following the proof of Theorem 3.1, we define

$$c_{1,k} = \inf_{\hat{\gamma} \in \Gamma_{1,k}} \max_{x \in D_1^{k+1}(0)} f(\hat{\gamma}(x)), \tag{3.11}$$

with

$$\Gamma_{1,k} = \{\hat{\gamma} \in C(D_1^{k+1}(0), E) \mid \hat{\gamma}(x) = \gamma(x),\ \forall\ x \in S^k\}. \tag{3.12}$$

As before, if $c_{1,k}$ is not a critical value of f, then, necessarily, $c_{1,k} = c$ and there exists $\hat{\gamma} \in \Gamma_{1,k} \cap f^c$. In this case, we define

$$c_{2,k} = \inf_{\hat{\gamma} \in \Gamma_{2,k}} \max_{x \in D_1^{k+1}(0)} f(\hat{\gamma}(x)), \tag{3.13}$$

with

$$\Gamma_{2,k} = \{\hat{\gamma} \in C(D_1(0), f^c) \mid \hat{\gamma}(x) = \gamma(x),\ \forall\ x \in S^k\}. \tag{3.14}$$

If $c_{2,k}$ is not a critical value of f, the argument employed in Lemma 3.4 shows that there exists $\hat{\gamma} \in \Gamma_{1,k}$ such that $\hat{\gamma}(D_1(0)) \subset S_c(f)$ and, consequently, $[\gamma] = 0$. Thus, we have shown that $[\gamma] \neq 0$ if, and only if, f has a critical value $d \neq c$. The theorem is proved. □

Remark 3.9. We observe that Theorem 3.8 may be used to establish new versions of generalizations of the Mountain Pass Theorem [**11, 12**].

4. Rolle's Theorem on $\mathbb{R}^2$

In this section, we apply Theorem 3.1 to establish our version of Rolle's Theorem on $\mathbb{R}^2$. Moreover, this theorem is applied to prove a global univalence result for a class vector fields on the plane. Before stating such results, we need to introduce some preliminaries.

Given $c \in \mathbb{R}^k$, and $F : \mathbb{R}^n \to \mathbb{R}^k$, $k \leq n$ of class C^1, we follow the notation used in section 2 and set $S_c(F) = F^{-1}(\{c\})$. We denote by K the set of singular points of F, and we note by K_c the set $K \cap S_c(F)$.

Theorem 4.1. (**Rolle's Theorem on** $\mathbb{R}^2$.) *Let $F = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field of class C^1 with f_1 satisfying (GPS) with respect to some $\phi \in \Lambda$. Assume that there exist two distinct points u and v belonging to $S_c(F)$, $c = (c_1, c_2)$, in such a way that c_1 is an admissible level for f_1. Then, either*

(i) *u and v are in the same path-component of $S_c(F)$, or*

(ii) *F has a singular value $d \neq c$.*

Remark 4.2. (i) Note that the above result is independent of the choice of the coordinates system, and that (GPS) condition is assumed for just one coordinate of F. (ii) We recall that the argument used in the proof of Theorem 4.2 also holds if f_1 is of class C^1 and f_2 is differentiable.

Proof. By Theorem 3.1, it suffices to suppose that u and v are in the same path-component of $S_{c_1}(f_1)$. Let $\gamma \in C([0,1], \mathbb{R}^2)$ be such that $\gamma(0) = u$, $\gamma(1) = v$, and

$$f_1(\gamma(t)) = c_1, \ \forall\ t \in [0,1]. \tag{4.1}$$

Then, $h = f_2 \circ \gamma \in C([0,1], \mathbb{R})$ and satisfies $h(0) = h(1) = c_2$. Furthermore, we have the two following excluding possibilities

(i) $h(t) = c_2$, for all $t \in [0,1]$, or

(ii) There exists $t \in (0,1)$ so that $h(t) \neq c_2$.

On the first case, u and v are in the same path-component of $S_c(F)$. If (ii) holds, we claim that F possesses a singular value $d \neq c$. Without loss of generality, we assume that $h(t_0) = \max_{t\in[0,1]} h(t) > c_2$. Arguing by contradiction, we suppose that the claim is not true. Since $\gamma(t_0)$ is a regular point of F, we may use (4.1) and the Implicit Function Theorem to assume that γ is differentiable on t_0 and $\gamma'(t_0) \neq 0$. From (4.1) and our choice of t_0, we get

$$(f_1'(\gamma(t_0)), \gamma'(t_0)) = 0.$$
$$(f_2'(\gamma(t_0)), \gamma'(t_0)) = 0.$$

Consequently, $f_1'(\gamma(t_0))$ and $f_2'(\gamma(t_0))$ are linearly dependent. Thus, $d = (c_1, f_2(\gamma(t_0))) \neq c$ is a singular value of F. The theorem is proved. □

Corollary 4.3. *Let $F = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field of class C^1 with f_1 satisfying (GPS). Assume that there exists $c = (c_1, c_2) \in \mathbb{R}^2$ such that F has no singular value $d \neq c$. Then, either*
(i) $S_c(F)$ possesses at most one point, or
(ii) $S_c(F)$ possesses a nontrivial component.

Proof. Let u and v be two distinct points of $S_c(F)$ and assume, by contradiction, that the components of $S_c(F)$ are trivial. Since F does not have a singular value $d \neq c$, we obtain that $\{w \in \mathbb{R}^2 \mid f_1'(w) = 0\} \subset S_c(F)$. Thus, c_1 is an admissible level of f_1. But, now Theorem 4.1 implies that F has a singular value $d \neq c$. The corollary is proved. □

Remark 4.4. We notice that we may substitute $S_c(F)$ by K_c in condition (ii).

Corollary 4.5. *Let $F = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field of class C^1 with f_1 satisfying (GPS) and $F(0) = 0$. Assume that F is a local diffeomorphism on $\mathbb{R}^2 \setminus S_0(F)$ and locally injective on $S_0(F)$. Then, $F(u) \neq 0$, for every $u \in \mathbb{R}^2 \setminus \{0\}$.*

Proof. If $F(u) = 0$ for some $u \neq 0$, by Corollary 4.3, $S_0(F)$ should have a nontrivial component. But, this contradicts the fact that F is locally injective on $S_0(F)$. The corollary is proved. □

Theorem 4.6. (**Global Injectivity Theorem.**) *Let $F = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field of class C^1 with f_1 satisfying (GPS) with respect to some $\phi \in \Lambda$. Then, F is globally injective provided it is a local diffeomorphism.*

Proof. Given $c \in \mathbb{R}^n$, $S_c(F)$ possesses only trivial components. Hence, Corollary 4.3 implies $S_c(F)$ has at most one point. Theorem 4.6 is proved. □

Remark 4.7. Theorem 4.6 holds for $F = (f_1, f_2)$ with f_1 of class C^1 and f_2 differentiable, if we assume that F is locally injective and has no singular points (See Remark 4.2-(ii)).

4.1. Examples

Here, we present some examples and applications of the results obtained in this section.

1. The following example has been given in [**6**]. Consider $F = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ the polynomial mapping defined by

$$F(x,y) = (x + (y + x^2)^2, y + x^2), \ \forall \ (x,y) \in \mathbb{R}^2.$$

Then, F is a local diffeomorphism since $\det(F'(x,y)) = 1$, for every $(x,y) \in \mathbb{R}^2$. Furthermore, it is easy to verify that $f_2(x,y)$ satisfies (PS), i. e., (GPS) with respect to $\phi(s) \equiv 1$. Therefore, by Theorem 4.6, F is globally injective.

We remark that we may not apply Theorem 1 of [**6**] to show that F is globally injective. We also notice that $\text{trace}(F'(x,y)) = 2 + 4x(y + x^2) \geq 2 > 0$ if $x \leq 0$ and $y \leq -x^2$, or $x \geq 0$ and $y \geq -x^2$. Hence, the global injectivity is not a consequence of the main result of [**3**].

2. Consider $F = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ of class C^1 such that $f_1(x,y) = \lambda x + e^x + \alpha(y)$, with $\lambda > 0$. Thus, $(f_1)_x(u) = \lambda + e^x > \lambda$, for every $u = (x,y) \in \mathbb{R}^2$. Assume that f_2 satisfies

$$(\lambda + e^x)(f_2)_y(u) - \alpha'(y)(f_2)_x(u) \neq 0, \ \forall \ u = (x,y) \in \mathbb{R}^2. \qquad (*)$$

Then, Theorem 4.6 implies that $F = (f_1, f_2)$ is globally injective. Note that we do not assume any extra hypothesis on $\det F'(x,y)$ besides condition (*).

3. Suppose that $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ is of class C^1 and assume that there exists $\phi \in \Lambda$ such that

(i) $\phi(\|(u)\|) \le \|f'(u)\|, \ \forall\, u \in \mathbb{R}^2$, and
(ii) $(f_x g_y - g_x f_y)(u) \neq 0, \ \forall\, u = (x,y) \in \mathbb{R}^2$.
Then, F is globally univalent in $\mathbb{R}^2$ by Theorem 4.6. This result is related to a theorem by Nikaido [**8**, **9**] which states that F is a global diffeomorphism under conditions stronger than (i) and (ii). We notice that it is not difficult to give examples of F satisfying (i) and (ii) which are not surjective (See the previous item).

4. Consider $F(x,y) = (f_1, f_2) = (e^x - y^2 + 3, 4ye^x - y^3)$. Then, F is a local diffeomorphism; but, it is not injective since $F(0,2) = F(0,-2) = (0,0)$. By Theorem 4.6, for every nonincreasing locally Lipschitz function $\phi : (0,\infty) \to (0,\infty)$ such that $\int_0^\infty \phi(t)\,dt = \infty$, there exists a sequence $(x_m, y_m) \to \infty$, as $m \to \infty$, such that $f_1(x_m, y_m) \to c_1 = 3$ and $\|f_1'(x_m,y_m)\|/\phi(\|(x_m,y_m)\| \to 0$, as $m \to \infty$. Note that the value $c_1 = 3$ is given by (3.1). A similar result holds for f_2.

5. Consider $f_1(x,y) = \log(x^2+1) - \log(y^2+1)$ and assume that $f_2 : \mathbb{R}^2 \to \mathbb{R}$ satisfies the following condition:

$$\frac{y}{y^2+1}(f_2)_x(u) + \frac{x}{x^2+1}(f_2)_y(u) \neq 0, \ \forall\, u = (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}. \quad (**)$$

Then, $F(x,y) \neq (0,0)$, for every $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Effectively, f_1 satisfies (GPS) with respect to $\phi(t) = t/(t^2+1)$ and $(**)$ implies F is a local diffeomorphism on $\mathbb{R}^2 \setminus \{(0,0)\}$. Thus, by Corollary 4.3, the result holds.

5. Rolle's Theorem on higher dimensions

In this section, we extend the results of the previous section to vector fields of class C^2 defined on dimensions greater than two.

Given $n \ge 3$ and $2 \le k \le n$, we denote by $\mathcal{P}_k = \{\sigma = \{j_1, \ldots, j_k\} \in \{1,\ldots,n\}^k \,|\, j_1 < \ldots < j_k\}$, the family of ordered subsets of $\{1,\ldots,n\}$ with k elements. For every $\sigma \in \mathcal{P}_k$, we consider the inclusion $j_\sigma : \mathbb{R}^k \to \mathbb{R}^n$ defined by

$$(j_\sigma(x))_l = \begin{cases} x_k, & l = j_k \in \sigma \\ 0, & l \notin \sigma. \end{cases}$$

We also take $P_\sigma : \mathbb{R}^n \to j_\sigma(\mathbb{R}^k)$ the corresponding projection and define

$\Phi_\sigma = j_\sigma^{-1} \circ P_\sigma : \mathbb{R}^n \to \mathbb{R}^k$. By $u_1 \wedge \ldots \wedge u_{k-1}$, we denote the vectorial product of $u_1, \ldots, u_{k-1}$ on $\mathbb{R}^k$. For $k = 2$ and $u = (a, b)$, we set $\wedge u = (-b, a)$.

Definition 5.1. *For $\sigma \in \mathcal{P}_k$, we define the σ-vectorial product of $v_1, \ldots, v_{k-1}$ on $\mathbb{R}^n$ by*

$$v_1 \wedge_\sigma \ldots \wedge_\sigma v_{k-1} = j_\sigma(\Phi_\sigma(v_1) \wedge \ldots \wedge \Phi\sigma(v_{k-1})).$$

It is not difficult to verify that $(v_1 \wedge_\sigma \ldots \wedge_\sigma v_{k-1}, v_i) = 0$, for every $i \in \{1, \ldots, k-1\}$.

Definition 5.2. *Given $v_i = (a_{i1}, \ldots, a_{in}) \in \mathbb{R}^n$, $1 \leq i \leq k \leq n$, and $\sigma = \{j_1, \ldots, j_k\} \in \mathcal{P}_k$, we consider the matrix $M_\sigma = (a_{ij_m})$, $1 \leq i, m \leq k$, and define*

$$\Delta_\sigma(v_1, \ldots, v_k) = \det\left[M_\sigma\right],$$

and

$$\Delta_k(v_1, \ldots, v_k) = \sqrt{\sum_{\sigma \in \mathcal{P}_k} \Delta_\sigma^2},$$

For $k = 1$ and $\sigma = \{j_1\}$, we take $\Delta_\sigma(v_1) = M_\sigma = a_{1j_1}$, and we obtain $\Delta_1(v_1) = \|v_1\|^2$. From the definitions given above and the properties of the vectorial product on $\mathbb{R}^k$, $k \geq 2$, we get

$$(v_1 \wedge_\sigma \ldots \wedge_\sigma v_{k-1}, v_k) = \Delta_\sigma(v_1, \ldots, v_k), \ \forall\, \sigma \in \mathcal{P}_k,$$
$$\Delta_{k-1}^2 = (n-k+1) \sum_{\sigma \in \mathcal{P}_k} \|v_1 \wedge_\sigma \ldots \wedge_\sigma v_{k-1}\|^2.$$

Now, we may define a generalization of (GPS) condition for vector fields $F = (f_1, \ldots, f_k) : \mathbb{R}^n \to \mathbb{R}^k$, $1 \leq k \leq n$. Given $m \in \{1, \ldots, k\}$, we denote by F_m the vector field $F_m = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$. Consider Λ, the family of functions given in section 2, and take $\phi \in \Lambda$. Setting $\|F'(u)\|_k = \Delta_k(f_1'(u), \ldots, f_k'(u))$, for every $u \in \mathbb{R}^n$, we define

Definition 5.3. *Given $F = (f_1, \ldots, f_k) : \mathbb{R}^n \to \mathbb{R}^k$, $2 \leq k \leq n$, be a vector field of class C^1 and $c \in \mathbb{R}^k$, we say that F satisfies the generalized Palais-Smale condition (with respect to ϕ) at c, denoted $(GPS)_c$, if every sequence $(u_m) \subset \mathbb{R}^n$ such that $F(u_m) \to c$ and*

$$\|F'(u_m)\|_k / \|F_{k-1}'(u_m)\|_{k-1} \phi(\|u_m\|) \to 0,$$

as $m \to \infty$, possesses a converging subsequence. If F satisfies $(GPS)_c$, for every $c \in \mathbb{R}^k$, we just say that it satisfies (GPS).

In the particular case where $k = 1$, we obtain the original (GPS) condition by taking $\|F_0'(u)\| \equiv 1$, for every $u \in \mathbb{R}^n$.

Following the notation used in section 1, we define $S_c(F) = F^{-1}(\{c\})$, for every $c \in \mathbb{R}^k$, and consider $K = \{u \in \mathbb{R}^n \mid \|F'(u)\|_k = 0\}$, the set of singular points of F. We also set $K_c = K \cap S_c(F) = \{u \in \mathbb{R}^n \mid F(u) = c,\ \|F'(u)\|_k = 0\}$, for every $c \in \mathbb{R}^k$. Note that if $K_c = \varnothing$, then $S_c(F)$ is a $n-k$ dimensional submanifold of R^n. In that case, we say that c is a regular value of F.

Definition 5.4. *Given $F \in C^1(\mathbb{R}^n, \mathbb{R}^k)$, we say that $c \in \mathbb{R}^k$ is an admissible value if either c is a regular value of F, or the components of K_c are trivial and c is an isolated singular value of F.*

Given $c \in \mathbb{R}^n$, we write $c = (a, b)$, $a \in \mathbb{R}^{k-1} \cong \mathbb{R}^{k-1} \times \{0\}$, $b \in \mathbb{R} \cong \{0\} \times \mathbb{R}$. The following result, also proved in section 6, is a natural generalization of Proposition 2.3.

Proposition 5.5. (Deformation Lemma.) *Suppose that $F = (f_1, \ldots, f_k) : \mathbb{R}^n \to \mathbb{R}^k \in C^2(\mathbb{R}^n, \mathbb{R}^k)$ and satisfies (GPS). Assume that (d, a) is the only possible critical value of F on $\{d\} \times [a, b)$ and that (d, a) is an admissible level for F. Then, there exists $\tau : [0,1] \times S_d(F_{k-1}) \cap (f_k^b \setminus K_{(d,b)}(F)) \to S_d(F_{k-1}) \cap (f_k^b \setminus K_{(d,b)}(F))$ such that*

(i) $\tau(0, u) = u,\ \forall\ u \in S_d(F_{k-1}) \cap (f_k^b \setminus K_{(d,b)}(F))$,

(ii) $\tau(t, u) = u,\ \forall\ (t, u) \in [0,1] \times S_{(d,a)}(F)$,

(iii) $\tau(1, u) \in S_{(d,a)}(F),\ \forall\ u \in S_d(F_{k-1}) \cap (f_k^b \setminus K_{(d,b)}(F))$.

Note that Proposition 5.5 establishes a deformation lemma for f_k on the manifold $M = S_d(F_{k-1})$ which may have singular points. The condition (GPS) on this case is based on the projection of the vector $\nabla f_k(u)$ over the tangent space of M at the point u.

Next, we establish the main results in this section. These are natural generalizations for vector fields of class C^2 of the results obtained in section 4.

Theorem 5.6. (Rolle's Theorem on $\mathbb{R}^n$, $n \geq 3$.) *Let $F = (f_1, \ldots, f_n) :$*

$\mathbb{R}^n \to \mathbb{R}^n$ *be a vector field of class* C^2 *with* $F_1, \ldots, F_{n-1}$ *satisfying (GPS) with respect to* $\phi_1, \ldots, \phi_{n-1} \in \Lambda$, *respectively. Assume that there exist two distinct points* u *and* v *belonging to* $S_c(F)$, $c = (c_1, c_2) \in \mathbb{R}^n$, *in such a way that* c_1 *is an admissible level for* F_{n-1}. *Then, either*

(i) u *and* v *are in the same path-component of* $S_c(F)$, *or*

(ii) F *has a critical value* $d \neq c$.

Proof. In the following, we use the notation $K(F) = \{u \in \mathbb{R}^n \mid \|F'(u)\| = 0\}$, $K_d(F) = K(F) \cap S_d(f)$, for every $F : \mathbb{R}^n \to \mathbb{R}^k$ and $d \in \mathbb{R}^k$. Considering $c = (c_1, c_2)$ given above, we set $c_1 = (c_1^1, \ldots, c_1^{n-1})$.

Next, we claim that u and v are in the same path-component of $S_{c_1}(F_{n-1})$ provided F has no singular value $d \neq c$. Observing that $K(F_k) \subset K(F_{n-1})$, we conclude that $\hat{c}_k = (c_1^1, \ldots, c_1^k)$ is an admissible value of F_k, for every $k \in \{1, \ldots, n-1\}$. Then, by Theorem 3.1, u and v are in the same path-component of $f_1 = F_1$.

Now, we argue by induction, assuming that u and v are in the same path-component of $S_{\hat{c}_{k-1}}(F_{k-1})$, for $2 \leq k \leq n-1$. Let $\gamma : [0,1] \to S_{\hat{c}_{k-1}}(F_{k-1})$ be a continuous path such that $\gamma(0) = u$ and $\gamma(1) = v$. Consider

$$b_k = \max_{t \in [0,1} f_k(\gamma(t)) \geq c_k.$$

If $b_k > c_k$, we apply Proposition 5.5 to obtain $\tau : [0,1] \times S_{\hat{c}_{k-1}}(F_{k-1}) \cap f_k^{b_k} \to S_{\hat{c}_{k-1}}(F_{k-1}) \cap f_k^{b_k}$ such that $\tau(1,u) \in S_{\hat{c}_{k-1}}(F_{k-1}) \cap f_k^{c_k}$, for every $u \in S_{\hat{c}_{k-1}}(F_{k-1}) \cap f_k^{b_k}$, and $\tau(t,u) = u$, for every $t \in [0,1]$, $u \in S_{\hat{c}_k}(F_k)$. Then, $\gamma_1 = \tau(1, \gamma(t)) : [0,1] \to S_{\hat{c}_{k-1}}(F_{k-1}) \cap f_k^{c_k}$ is a continuous path such that $\gamma_1(0) = u$ and $\gamma_1(1) = v$. Taking $\gamma_1 = \gamma$ if $b_k = c_k$, we define

$$a_k = \min_{t \in [0,1]} f_k(\gamma_1(t)) \leq c_k.$$

If $a_k = c_k$, we have the claim. Otherwise, we may argue as above and apply Proposition 5.5 to $\hat{F}_k = (f_1, \ldots, -f_k)$ to obtain that u and v are in the same path-component of $S_{\hat{c}_k}(F_k)$. This proves the claim.

So it suffices to assume that u and v are in the same path-component of $S_{c_1}(F_{n-1})$. Let $\gamma : [0,1] \to S_{c_1}(F_{n-1})$ be a continuous path such that $\gamma(0) = u$ and $\gamma(1) = v$. Take $h = f_n \circ \gamma : [0,1] \to \mathbb{R}$. As in the proof of Theorem 4.1, we have the following two excluding possibilities

(i) $h(t) = c_n$, for all $t \in [0, 1]$, or
(ii) There exists $t \in (0, 1)$ so that $h(t) \neq c_n$.

On case (i), u and v are in the same path-component of $S_c(F)$. If (ii) holds, we use the same argument of the proof of Theorem 4.1 to conclude that F has a singular value $d \neq c$. Theorem 5.6 is proved. □

We omit the proof of the next three results since the argument employed is similar to the one used to prove the corresponding results in section 4.

Corollary 5.7. *Let $F = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field of class C^2 with $F_1, \ldots, F_{n-1}$ satisfying (GPS) with respect $\phi_1, \ldots, \phi_{n-1} \in \Lambda$, respectively. Assume that there exists $c \in \mathbb{R}^n$ such that F has no singular value $d \neq c$. Then, either*
(i) *$S_c(F)$ possesses at most one point, or*
(ii) *$S_c(F)$ possesses a nontrivial component.*

Corollary 5.8. *Let $F = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field of class C^2 with $F(0) = 0$ and $F_1, \ldots, F_{n-1}$ satisfying (GPS) with respect to $\phi_1, \ldots, \phi_{n-1} \in \Lambda$, respectively. Assume that F is a local diffeomorphism on $\mathbb{R}^n \setminus S_0(F)$ and locally injective on $S_0(F)$. Then, $F(u) \neq 0$, for every $u \in \mathbb{R}^n \setminus \{0\}$.*

Theorem 5.9. **(Global Injectivity Theorem.)** *Let $F = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field of class C^2 with $F_1, \ldots, F_{n-1}$ satisfying (GPS) with respect to $\phi_1, \ldots, \phi_{n-1} \in \Lambda$, respectively. Then, F is globally injective provided it is a local diffeomorphism.*

The next result is an immediate application of Theorem 5.6

Theorem 5.10. *Let $F = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field of class C^2 with $f_1, \ldots, f_{n-1}$ satisfying (GPS) with respect to $\phi_1, \ldots, \phi_{n-1} \in \Lambda$, respectively. Assume that F is a local diffeomorphism on $\mathbb{R}^n$. Then, F is globally injective provided it satisfies*
(i) *For each $k \in \{2, \ldots, n-1\}$, there exist $\theta_k > 0$ and $R_k > 0$ such that*

$$\|F_k'(u)\|_k \geq \theta_k \|f_k'(u)\| \|F_{k-1}'(u)\|_{k-1}, \ \forall \ \|u\| \geq R_k.$$

5.1. Examples

Next, we illustrate Theorem 5.9, presenting some examples and applications.

1. Consider $F = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ of class C^2 such that F is a local diffeomorphism, $f_1(u) = f(x_1, x_2)$, for every $u = (x_1, x_2, x_3) \in \mathbb{R}^3$, and f_1 satisfies (GPS). Then, F is globally injective provided there exists $\phi \in \Lambda$ so that $\|(f_2)_{x_3}(u)\| \geq \phi(\|u\|)$, for every $u \in \mathbb{R}^3$. Indeed, by Theorem 5.9 it suffices to show that F_2 satisfies (GPS). But, by definition,

$$\begin{aligned}\|F_2'(u)\|_2^2 &\geq \Delta_{\{1,3\}}^2(f_1'(u), f_2'(u)) + \Delta_{\{2,3\}}^2(f_1'(u), f_2'(u)) \\ &\geq \|f_1'(u)\|^2\phi^2(\|u\|),\ \forall\ u \in \mathbb{R}^3.\end{aligned}$$

Hence, F is globally injective. As a direct application of this result, we obtain, for example, that $F(x,y,z) = (x+(y+x^2)^2+z, y+x^2, x/2+z^3+z)$ is univalent on $\mathbb{R}^3$.

2. Given $\epsilon \in \mathbb{R}$, Consider $F_\epsilon = (f_\epsilon, g, h) : \mathbb{R}^3 \to \mathbb{R}^3$ of class C^2 such that $f_\epsilon(x,y,z) = \epsilon x + e^{2x} + \log(y^2+1)$, $g(x,y,z) = ye^{-2x} + g_1(z)$, $h(x,y,z) = h_1(y,z)$ with $(h_1)_z(y,z) > 0$ and $g_1'(z)(h_1)_y(y,z) \leq 0$ for every $(x,y,z) \in \mathbb{R}^3$. We determine the injectivity of F_ϵ in function of the parameter ϵ:

(a) F_ϵ is globally injective if $\epsilon > 0$. Since $f_\epsilon'(x,y,z) = (\epsilon + 2e^{2x}, 2y, 0)$, we have that f_ϵ satisfies (PS). Furthermore, we claim that $(F_\epsilon)_2$ also satisfies (PS), i.e., (GPS) with respect to $\phi(t) \equiv 1$. We argue by contradiction one more time, supposing that there exists a sequence $u_m = (x_m, y_m, z_m)$, $m \in \mathbb{N}$, such that $(F_\epsilon)_2(u_m) \to c = (c_1, c_2)$, $\|(F_\epsilon)_2'(u_m)\|_2/\|f_\epsilon'(u_m)\| \to 0$, and $\|u_m\| \to \infty$, as $m \to \infty$. From $f_\epsilon(u_m) \to c_1$, we conclude that either $(x_m, y_m) \in \mathbb{R}^2$ is a bounded sequence or $(x_m, y_m) \to (-\infty, \infty)$, as $m \to \infty$. On both cases $\|f_\epsilon'(u_m)\| \to d \neq 0$ and, consequently, $\|(F_\epsilon)_2'(u_m)\| \to 0$. But, this contradicts

$$\|(F_\epsilon)_2'(u)\|_2 \geq \Delta_{\{1,2\}}(f_\epsilon', g')(u) = \epsilon e^{-2x} + 4y^2e^{-2x}/(y^2+1) + 2 \geq 2,$$

for $u = (x,y,z) \in \mathbb{R}^3$. The claim is proved. Since F_ϵ is a local diffeomorphism on $\mathbb{R}^3$, Theorem 5.9 implies F_ϵ is globally univalent.

(b) F_ϵ is not injective if $\epsilon < 0$. Let $x_\epsilon \in \mathbb{R}$ be such that $\epsilon x_\epsilon + e^{2x_\epsilon} = 1$. Then, $F_\epsilon(x_\epsilon, 0, 0) = F_\epsilon(0,0,0) = (1, g_1(0), h(0,0))$.

(c) F_ϵ is globally injective if $\epsilon = 0$. Let $u_m = (x_m, y_m, z_m)$, $m \in \mathbb{N}$, be a sequence such that $f_0(u_m) \to c_1 > 0$. By the definition of f_0, we find $M > 0$ such that $x_m \leq M$, $|y_m| \leq M$, for every $m \in \mathbb{N}$. Furthermore, $(x_m, y_m) \not\to (-\infty, 0)$ since, on that case $f_0(u_m) \to 0$, as $m \to \infty$. Using those facts, we obtain that $\|f_0'(u_m)\| \to d \neq 0$, as $m \to \infty$. Now, we may argue as on item (a) to conclude that f_0 satisfies $(PS)_c$, for every $c > 0$, and $(F_0)_2$ satisfies $(PS)_c$, for every $c = (c_1, c_2)$ with $c_1 > 0$.

As $f_0(u) > 0$, for every $u \in \mathbb{R}^3$, we deduce from Theorem 3.1 and Remark 3.5-(i) that $S_c(f_0)$ is path-connected for every $c > 0$. Actually, the same argument allows to apply Theorem 5.9 to conclude that F_0 is globally injective since $(F_0)_2$ satisfies $(PS)_c$, for $c = (c_1, c_2)$, with $c_1 > 0$.

6. Proofs of Propositions 2.3 and 5.5

Proof of Proposition 2.3. Let $V(u)$ be a pseudo-gradient vector field associated to f on $\tilde{E} = \{u \in E \,|\, f'(u) \neq 0\}$ (See [**11**]). Recall that V is a locally Lipschitz continuous map so that, for every $u \in \tilde{E}$, we have

$$\begin{aligned} &\|V(u)\| \leq 2\|f'(u)\|, \\ &(V(u), f'(u)) \geq \|f'(u)\|^2. \end{aligned} \tag{6.1}$$

Given the initial value problem

$$\begin{aligned} \frac{d}{dt}\eta(t,u) &= -\frac{V(\eta(t,u))\phi(\|\eta(t,u)\|)}{\|V(\eta(t,u))\|^2}, \\ \eta(0,u) &= u \in A \equiv f^b \setminus (K_b \cup f^a), \end{aligned} \tag{6.2}$$

By the theorem of existence and uniqueness for ordinary differential equations, there exists a maximum time $t^+(u) > 0$ such that $\eta(t,u)$ is defined on $[0, t^+(u))$, for $u \in A$. Setting $T_u = \max\{s \in [0, t^+(u)), \,|\, f(\eta(s,u) > a\}$, for $u \in A$, our main objective is to study the behaviour of $\eta(t,u)$ as $t \to T_u$.

Using the properties (6.1) of a pseudo-gradient vector field, we have

$$\frac{d}{dt} f(\eta(t,u)) \leq -\frac{1}{4}\phi(\|\eta(t,u)\|) < 0, \ \forall\, t \in [0, t^+(u)). \tag{6.3}$$

Furthermore, given $d > 0$, by (GPS), we obtain $\delta > 0$ such that

$$\|f'(u)\| \geq \delta\phi(\|(u)\|), \ \forall\ u \in A \setminus N_d(K). \tag{6.4}$$

Assuming the hypothesis and the notation of Proposition 2.3, we need some preparatory lemmas.

Lemma 6.1. *There exists $d_0 > 0$ such that for every solution $\eta(.,u)$ of (6.2), $0 < t_1 < t_2 < T_u$ and $0 < d \leq d_0$ satisfying*

$$\begin{aligned} &\|\eta(t_1,u) - K_a\| = d/2, \\ &\|\eta(t_2,u) - K_a\| = d, \\ &d/2 \leq \|\eta(t,u) - K_a\| \leq d, \ \forall\ t_1 \leq t \leq t_2. \end{aligned} \tag{6.5}$$

Then, there exist $\delta = \delta(d) > 0$ and $\alpha = \alpha(d) > 0$ such that

$$t_2 - t_1 \geq \frac{\delta d}{2}, \tag{6.6}$$

and

$$a < f(\eta(t_2,u)) \leq f(\eta(t_1,u)) - \frac{1}{4}\alpha(t_2 - t_1). \tag{6.7}$$

Proof. Taking d_0 smaller if necessary, we choose $\delta = \delta(d)$ such that (6.4) holds for every $u \in A \setminus N_{d/2}(K)$. Hence,

$$\|\frac{d}{dt}\eta(t,u)\| \leq \frac{1}{\delta}, \ \forall\ t \in [t_1,t_2].$$

That implies immediately (6.6). Furthermore, since K_a is compact, from (6.5), we get $M > 0$ such that $\|\eta(t,u)\| \leq M$, for every $t \in [t_1,t_2]$. Using (6.3), we obtain $\alpha > 0$ so that

$$\frac{d}{dt}f(\eta(t,u)) \leq -\frac{1}{4}\alpha, \ \forall\ t \in [t_1,t_2].$$

The above inequality provides (6.7). So Lemma 6.1 is proved. □

Definition 6.2. *Given a topological space M, a compact set $K \subset M$, and a continuous map $\eta : [0,t_0) \to M$, $t_0 \in \mathbb{R} \cup \{\infty\}$, we say that the pair (η, K) satisfies the strong attraction property, denoted (SAP), at t_0 if there exists $v \in K$ such that $\lim_{t\to t_0} \eta(t) = v$, whenever $\liminf_{t\to t_0} \|\eta(t) - K\| = 0$.*

Lemma 6.3. *Given $u \in A$ and considering $\eta(t,u)$ the corresponding solution of (6.2), we have*
(a) $f(\eta(t,u)) \to a$ as $t \to T_u$,
(b) The pair $(\eta(.,u), K_a)$ satisfies (SAP) at T_u.

Proof. Arguing by contradiction, we suppose that (a) does not occur. From (6.3), there exists $\hat{a} \in (a,b)$ such that $f(\eta(t,u)) \to \hat{a}$, as $t \to T_u$. Using the compactness of $K_a \cup K_b$, we obtain $d > 0$ such that $\eta(t,u) \in A \setminus N_d(K)$ for every $t \in [0, T_u)$. Consequently, by (6.4), we find $\delta > 0$ so that $\|f'(\eta(t,u))\| \geq \delta\phi(\|\eta(t,u)\|)$, $t \in [0,T_u)$. The last inequality and (6.2) imply that $\|\frac{d}{dt}\eta(t,u)\| \leq \frac{1}{\delta}$, for every $t \in [0,T_u)$. Therefore, $\|\eta(t,u)\| \leq \|u\| + \frac{t}{\delta}$, and

$$f(\eta(t,u)) \leq f(u) - \int_0^t \phi(\|u\| + \frac{s}{\delta})\, ds,\ \forall\, t \in [0,T_u),$$

provided that $\phi(s)$ is nonincreasing. As $\int_0^\infty \phi(s)\,ds = \infty$, we get that $T_u < \infty$ and, from the above bound on $\|\frac{d}{dt}\eta(t,u)\|$, there exists $v \in S_{\hat{a}}(f)$ such that $\eta(t,u) \to v$, as $t \to T_u$. But, that contradicts the definition of T_u since $\eta(t,u)$ would be defined on $[0,s]$, $s > T_u$, and $f(\eta(t,u)) > a$, for every $t \in [0,s]$. That proves (a).

To verify (b), we suppose $\liminf_{t\to T_u} \|\eta(t,u) - K_a\| = 0$. First of all, we assert that $\lim_{t\to T_u} \|\eta(t,u) - K_a\| = 0$. Arguing by contradiction, we assume that this is not verified. Considering d_0 given by Lemma 6.1, we find $0 < d \leq d_0$ and sequences $0 < t_1 < s_1 < \ldots < t_m < s_m < \ldots < T_u$ such that t_m and s_m satisfy (6.5), for every $m \in \mathbb{N}$. Consequently, $s_m - t_m \geq \frac{\delta d}{2}$, and

$$a < f(\eta(s_m,,u)) \leq f(\eta(t_m,u)) - \frac{1}{4}\alpha(s_m - t_m) \leq f(\eta(s_{m-1},u)) - \frac{1}{8}\alpha d\delta.$$

Hence, $T_u < \infty$ and $s_m - t_m \to 0$, as $m \to \infty$. But, that provides a contradiction. Therefore, $\lim_{t\to T_u} \|\eta(t,u) - K_a\| = 0$, as claimed.

From the compactness of K_a, we get $M < \infty$ such that $\|\eta(t,u)\| \leq M$, for every $t \in [0,T_u)$. Using this relation and (6.3), we get $\hat{\alpha} > 0$ so that

$$\frac{d}{dt} f(\eta(t,u)) \leq -\hat{\alpha},\ \forall\, t \in [0,T_u).$$

This equation implies that $T_u < \infty$. Furthermore, since K_a is compact, we may find $z_1 \in K_a$ and a sequence $\hat{t}_m \to T_u$, as $m \to \infty$, so that $\eta(\hat{t}_m, u) \to z_1$, as $m \to \infty$. If $\eta(t,u) \not\to z_1$, as $t \to T_u$, we would obtain $z_2 \in K_a \setminus \{z_1\}$ and $\hat{s}_m \to T_u$ satisfying $\eta(\hat{s}_m, u) \to z_2$, as $m \to \infty$. As the connected components of K_a are points, by the Separation Lemma [**2**], there exist disjoint compact sets K_1 and K_2 satisfying $K_a = K_1 \cup K_2$ and $z_i \in K_i$, $i = 1, 2$. Taking $0 < d < \min\{d_0, \frac{1}{2}\|K_1 - K_2\|\}$, we find sequences $0 < t_1 < s_1 < \ldots < t_m < s_m < \ldots < T_u$, with $t_m \to T_u$, as $m \to \infty$, and t_m, s_m satisfying (6.5), for every $m \in \mathbb{N}$. The same argument used above gives us a contradiction. The lemma is proved. □

The reasoning employed in the proved of Lemma 6.3 provides a uniform bound for $\eta(t,u)$ on the interval $[0, T_u)$.

Corollary 6.4. $T_u < \infty$, *for every* $u \in A$. *Moreover, there exists* $M < \infty$ *such that* $\|\eta(t,u)\| \leq M$, *for every* $t \in [0, T_u)$.

Proof. If $\liminf_{t \to T_u} \|\eta(t,u) - K_a\| = 0$, The thesis of Corollary 6.4 has already been proved in Lemma 6.3. Otherwise, we argue as in the first part of the proof of that lemma to obtain the conclusion. □

Lemma 6.5. (a) *Given* $u \in A$, *there exists* $v \in S_a$ *so that* $\lim_{t \to T_u} \eta(t,u) = v$.

(b) *The application* $T : A \to \mathbb{R}$ *given by* $T(u) = T_u$ *is a well defined continuous map.*

(c) *Given* $u \in S_a$ *and a sequence* $(u_m) \subset A$ *such that* $u_m \to u$, *as* $m \to \infty$, *we have* $T(u_m) \to 0$, *as* $m \to \infty$.

Proof. To show (i)-(ii), we consider two possibilities:

1. $\liminf_{t \to T_u} \|\eta(t,u) - K_a\| \geq d > 0$. By Corollary 6.4, $T_u < \infty$ and $\eta(t,u)$ is uniformly bounded for $t \in [0, T_u)$. Applying (6.4) and the argument used in Lemma 6.3, we get that $\eta(t,u) \to v \in S_a(f) \setminus K_a$. (ii) is a direct consequence of continuous dependence of solutions of (6.2) with respect to initial conditions.

2. $\liminf_{t \to T_u} \|\eta(t,u) - K_a\| = 0$. Lemma 6.3 implies the pair $(\eta(t,u), K_a)$ satisfies (SAP). Thus, (i) must hold. Furthermore, by Corollary 6.4,

$T_u < \infty$. To show that $T(u)$ is continuous on A, we argue by contradiction, supposing there exist $\gamma > 0$ and $(u_m) \subset A$ such that $u_m \to u \in A$ and $|T_{u_m} - T_u| \geq \gamma > 0$, for every $m \in \mathbb{N}$. From the continuity of $\eta(T_u - \gamma, .)$ at u, without loss of generality, we may suppose $T_{u_m} \geq T_u + \gamma$, for every $m \in \mathbb{N}$.

We claim that there exists $\hat{a} \in (a, \min\{b, f(u)\})$ satisfying $f(\eta(T_u + \gamma/2, u_m)) > \hat{a} > a$, for every $m \in \mathbb{N}$.

Assuming the claim, we take $\hat{T}_u = \max\{s \in [0, t(u)) \,|\, f(\eta(s, u)) > \hat{a}\}$, $u \in I^b \setminus (K_b \cup I^{\hat{a}})$. By our previous argument, $\hat{T}_u$ is continuous since $K_{\hat{a}} = \varnothing$. Furthermore, $\hat{T}_{um} \geq T_u + \gamma/2$, for every $m \in \mathbb{N}$. Hence, $\hat{T}_u \geq T_u + \gamma/2$ because $u_m \to u$, as $m \to \infty$. But, that contradicts the definition of T_u. Thus, to prove the continuity of T_u, it suffices to verify the claim. By the continuity fo f, the fact that K_b is compact and (6.4), there exists $d_1 > 0$ such that $\|\eta(t, u_m) - K_b\| \geq d_1$, for every $m \in \mathbb{N}$ and $t \in (0, t_{u_m})$. Considering $d_0 > 0$ given by Lemma 6.1, we take $d_2 = \min\{d_1, d_0\}$. Taking γ smaller, if necessary, we may assume that $\gamma < 3\delta(d_2)d_2$, with $\delta(d_2)$ given by Lemma 6.1, and that $\eta(., u_m)$ is well defined on $I = [T_u - \gamma/2, T_u + 2\gamma/3]$, for every $m \in \mathbb{N}$.

Now, we consider the three possibles cases:

(a) $\limsup_{m\to\infty} \|\eta(I, u_m) - K_a\| \leq d_2$. Since K_a is compact, we obtain $M < \infty$ such that $\|\eta(t, u_m)\| \leq M$, for every $t \in I$, $m \in \mathbb{N}$. Applying (6.3) and observing that $f(\eta(T_u + 2\gamma/3, u_m) \geq a$, we obtain the claim.

(b) $\liminf_{m\to\infty} \|\eta(I, u_m) - K_a\| \geq d_2/2 > 0$. In view of (6.4), we get

$$\|\frac{d}{dt}\eta(t, u_m)\| \leq \frac{1}{\delta}, \ \forall\, t \in I, \ m \in \mathbb{N}.$$

Therefore,

$$\|\eta(t, u_m)\| \leq \|\eta(T_u - \gamma/2, u)m)\| + \frac{1}{\delta}(t - T_u + \gamma/2), \ \forall\, t \in I, \ m \in \mathbb{N}.$$

From the continuity of $\eta(T_u - \gamma/2, .)$ at u, we also obtain that $\|\eta(t, u_m)\|$ is unformly bounded for $t \in I$, $m \in \mathbb{N}$. As before the claim is proved.

(c) $\liminf_{m\to\infty} \|\eta(I, u_m) - K_a\| < d_2/2 < d_2 < \limsup_{m\to\infty} \|\eta(I, u_m) - K_a\|$. Taking a subsequence if necessary, we find $t_m < s_m$, $t_m, s_m \in I$

such that (6.5) holds. Applying Lemma 6.1, we have

$$f(\eta(T_u + \gamma/2, u_m)) \geq f(\eta(s_m, u)) + \frac{1}{8}\alpha d\delta > a, \ \forall \ m \in \mathbb{N}.$$

That proves the claim.

Our final task is to verify that $T_{um} \to 0$, as $u_m \to u \in S_a(f)$. If $u \notin K_a$, the result is easily obtained.

Thus, we may suppose $u \in K_a$. Arguing by contradiction, we assume $|T_{um}| \geq \epsilon$, for every $m \in \mathbb{N}$. If there exists $M < \infty$ such that $\|\eta(t, u_m)\| \geq M$, for every $t \in [0, T_{um})$, $m \in \mathbb{N}$, from (6.3) and $f(u_m) \to f(u)$, we must have $T_{um} \to 0$. Hence, The last relation cannot happen. As $u_m \to u$, we obtain $0 < d \leq d_0$ and $t_m < s_m$, t_m, $s_m \in [0, T_{um})$ such that (6.5) holds for every $m \in \mathbb{N}$. Using Lemma 6.3 and $f(u_m) \to f(u)$, we also obtain a contradiction. Lemma 6.5 is proved. □

Using Lemma 6.5, we set $\eta(T_u, u) = \lim_{t \to T_u} \eta(t, u)$, for every $u \in A$.

Lemma 6.6. *The application* $\eta_1 : [0,1] \times A \to f^b \backslash K_b$ *defined by* $\eta_1(t, u) = \eta(tT_u, u)$ *is continuous. Moreover, if* $(t_m, u_m) \in [0,1] \times A$, *for every* $m \in \mathbb{N}$, *and* $u_m \to u \in S_a(f)$, *as* $m \to \infty$; *then,* $\eta_1(t, u) \to u$, *as* $m \to \infty$.

Proof. The continuity of η_1 at $(t, u) \in [0,1] \times A$ when $\eta(T_u, u) \notin K_a$ or $t < 1$ is easily obtained.

Thus, we suppose $v = \eta(T_u, u) \in K_a$ and $t = 1$. Arguing by contradiction once again, we assume that there exists $((t_m, u_m)) \subset [0,1] \times A$ so that $u_m \to u$, $t_m \to 1$, but $v_m = \eta(t_m T_{um}, u_m) \not\to v$, as $m \to \infty$. Furthermore, we may assume there exists $\epsilon > 0$ such that $\|\eta(t_m T_{um}, u_m) - v\| \geq \epsilon > 0$, for every $m \in \mathbb{N}$. Now, we consider the following possibilities:

1. $\|\eta(t_m T_{um}, u_m) - v\| \geq d > 0$. Since the pair $(\eta(t, u), K_a)$ satisfies (SAP) at T_u, we have $\gamma > 0$ such that $\|\eta(t, u) - K_a\| \leq \frac{d}{4}$, for every $t \in [T_u - \gamma, T_u]$. The continuity of T_u provides the existence of $m_0 \in \mathbb{N}$ such that $t_m T_{um} \geq T_u - \gamma/2$, for every $m \geq m_0$. Moreover, $\eta(T_u - \gamma, u_m) \to \eta(T_u - \gamma, u)$, as $m \to \infty$. Taking d smaller if necessary, we obtain $T_u - \gamma < \hat{t}_m < \hat{s}_m < T_{um}$ such that (6.5) holds

for m sufficiently large. Invoking Lemma 6.1, we get δ_0, independent of m, so that $\delta_0 d < 2(\hat{s}_m - \hat{t_m}) \leq T_{um} - T_u + \gamma$. As $T_{um} \to T_u$, as $m \to \infty$, and $\gamma > 0$ can be taken arbitrarily small, we get a contradiction.

2. $\liminf_{m\to\infty} \|\eta(t_m T_{um}, u_m) - K_a\| = 0$. By the compactness of K_a, we may assume $\eta(t_m T_{um}, u_m) \to \hat{v} \in K_a \setminus \{v\}$. Now, we use the separation lemma and the argument used before to get a contradiction with $T_{um} \to T_u$, as $m \to \infty$. That shows the continuity of η_1.

Working in a similar fashion, we may also verify that $\eta(t_m T_{um}, u_m) \to u$ when $u_m \to u$, as $m \to \infty$. Lemma 6.6 is proved. □

Conclusion of the Proof of Proposition 2.3. Define $\tau : [0,1]\times(f^b\setminus K_a) \to (f^b \setminus K_b)$ by

$$\tau(t,u) = \begin{cases} u, & (t,u) \in [0,1] \times f^a, \\ \eta_1(t,u), & (t,u) \in [0,1] \times f^b \setminus (K_b \cup f^a). \end{cases}$$

From Lemmas 6.5 and 6.6, $\tau(t,u)$ is a continuous map and satisfies the thesis of Proposition 2.3. □

Remark 6.7. We may have $b = \infty$ on Proposition 2.3. In other words, if the functional $f(u)$ does not have a critical value above the level a; then, f^a is a strong deformation retract of E.

Proof of Proposition 5.5. The proof is similar to the proof of Proposition 2.3. For that reason, we only verify the main estimates.

First, we consider the vector field $W : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$W(u) = \begin{cases} \dfrac{\sum_{\sigma\in\mathcal{P}_k}(f_1'(u)\wedge\sigma\ldots\wedge\sigma f_{k-1}'(u))\Delta_\sigma(u)}{\|F'(u)\|_k^2}, & \text{if } u \notin K, \\ 0, \text{ otherwise}, \end{cases}$$

where we used the notation $\Delta_\sigma(u) = \Delta_\sigma(f_1'(u), \ldots, f_{k-1}'(u))$. Next, we consider the following initial value problem

$$\begin{aligned} \frac{d}{dt}\eta(t,u) &= -W(\eta(t,u))\phi(\|\eta(t,u)\|) \\ \eta(0,u) &= u \in A \equiv S_d(F_{k-1}) \cap f_k^b \setminus (K_{(d,b)} \cup f_k^a). \end{aligned} \tag{6.8}$$

For every $u \in A$, $\eta(t,u)$ is defined on a maximal interval $[0, t^+(u))$. Given $t \in [0, t^+(u))$, we get

$$\frac{d}{dt} f_i(\eta(t,u)) = -(f_i'(\eta(t,u)), W(\eta(t,u)))\phi(\|\eta(t,u)\|), \ \forall\, i \in \{1, \ldots, k\}.$$

Therefore, by the definition of W, we have

$$\frac{d}{dt} f_i(\eta(t,u)) = 0, \ \forall\, t \in [0, t^+(u)), \ i \in \{1, \ldots, k-1\}.$$

This implies that $\eta(t,u) \in S_d(F_{k-1})$, for every $t \in [0, t^+(u))$. Moreover, for every $t \in [0,1]$,

$$\begin{aligned} \frac{d}{dt} f_k(\eta(t,u)) &= -\frac{\sum_{\sigma \in \mathcal{P}_k} \Delta_\sigma^2(\eta(t,u))}{\|F'(\eta(t,u))\|_k^2} \phi(\|\eta(t,u)\|) \\ &= -\phi(\|\eta(t,u)\|). \end{aligned} \tag{6.9}$$

By (GPS), given $d > 0$, we get $\delta > 0$ such that

$$\|F'(u)\|_k \geq \delta\phi(\|u\|)\|F_{k-1}'(u)\|_{k-1}, \ \forall\, u \in A \setminus N_\delta(K). \tag{6.10}$$

Consequently, by Schwarz inequality on $\mathbb{R}^N$, where $N = \#(\mathcal{P})_k$, we get

$$\begin{aligned} W(u) &\leq \left(\sum_{\sigma \in \mathcal{P}_k} \|f_1'(u) \wedge_\sigma \ldots \wedge_\sigma f_{k-1}'(u)\|^2 \right)^{1/2} \|F'(u)\|_k^{-1} \\ &\leq (n-k+1)^{1/2} \|F_{k-1}'(u)\|_{k-1} \|F'(u)\|_k^{-1} \\ &\leq (n-k+1)^{1/2} (\delta\phi(\|u\|))^{-1}, \end{aligned} \tag{6.11}$$

for every $u \in A \setminus N_\delta(K)$. It is not difficult to see that equations (6.9), (6.10), (6.11) and the argument used in Proposition 2.3 provide the proof of Proposition 5.5. □

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